Constructive Output of Existentially Proved Structure in Combinatorics

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Based on several works with Xi Chen, Edmonds, Feng, Kulkarni, Liu, Papadimitriou, Qi, Xu

Outline

一、Parity Arguments for ∃

二、Problems in PPAD

三、PPAD-Completeness

四、2 Player Nash

五、Octahedral TUCKER

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Another End of Undirected Lines (AEUL)

Given an undirected graph G=(V,E;s) of degree no more than 2 with a degree-one node s
There exists another node t of degree-one.

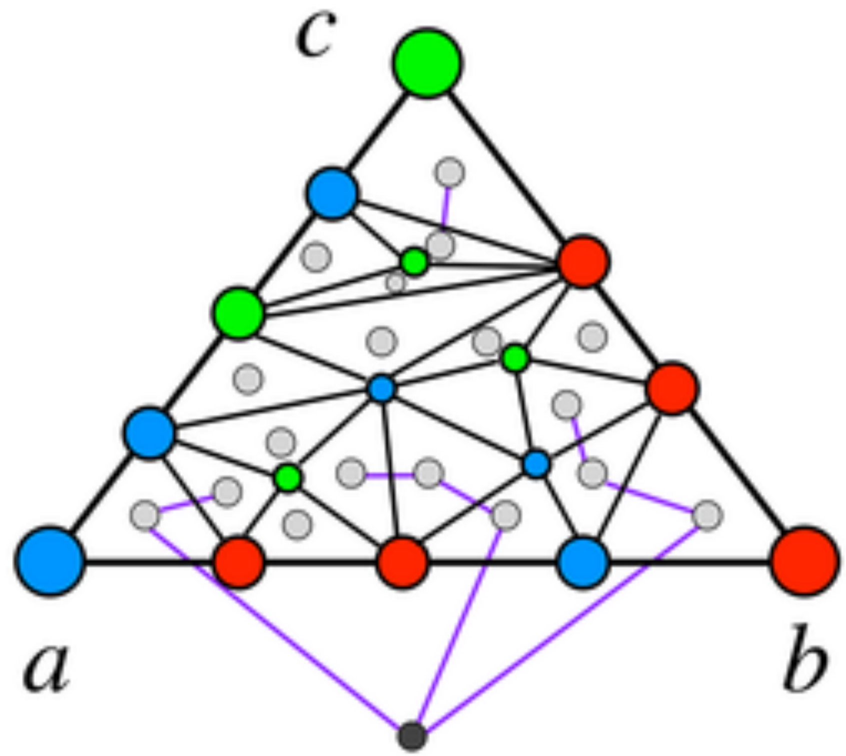
Examples:

1. The Sperner Lemma
2. The Smith Theorem

Sperner Lemma

Given a triangle ABC & its triangulation *T*.
Base triangle: the minimal size triangles in T.

Sperner Coloring of the set *S* of vertices of *T*:
1. A, B, and C are colored Blue, red, and green respectively
2. Each vertex on an edge of ABC is to be colored only with one of the two colors of the ends of its edge.
   - E.g., each vertex on AC must have a color either blue or green.



Sperner triangle: A triangle from *T*, with all three different colors.
- Lemma: there must be an odd number of **Sperner triangle**s.

SPERNER: Boundary vertices are so colored that each edge has one color internally

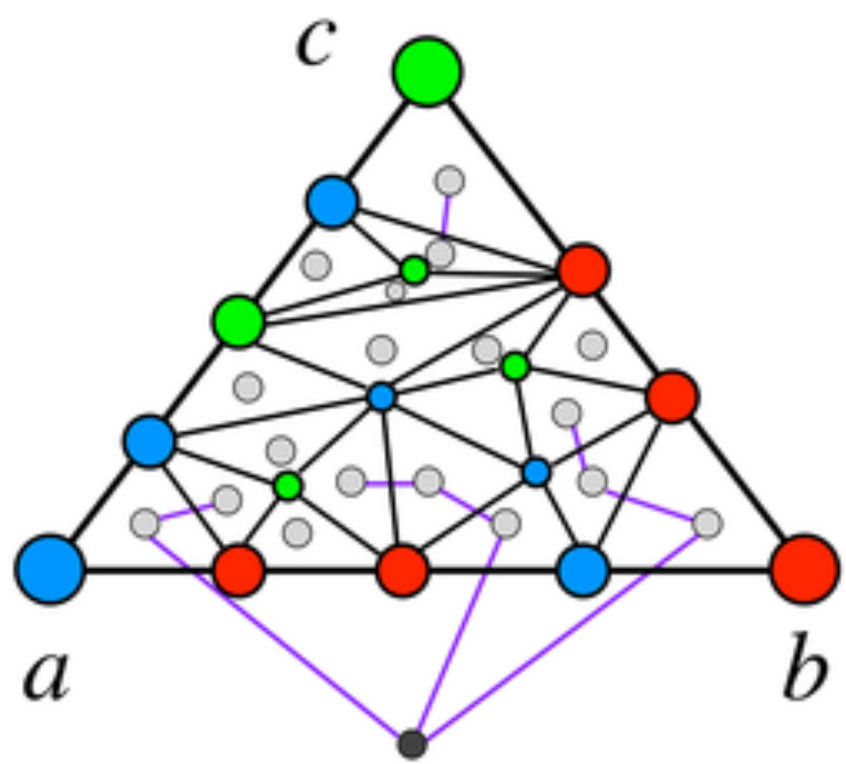
The underlying graph

Nodes: base triangles of T

Edges: between two nodes if they share a boundary edge colored by blue and red

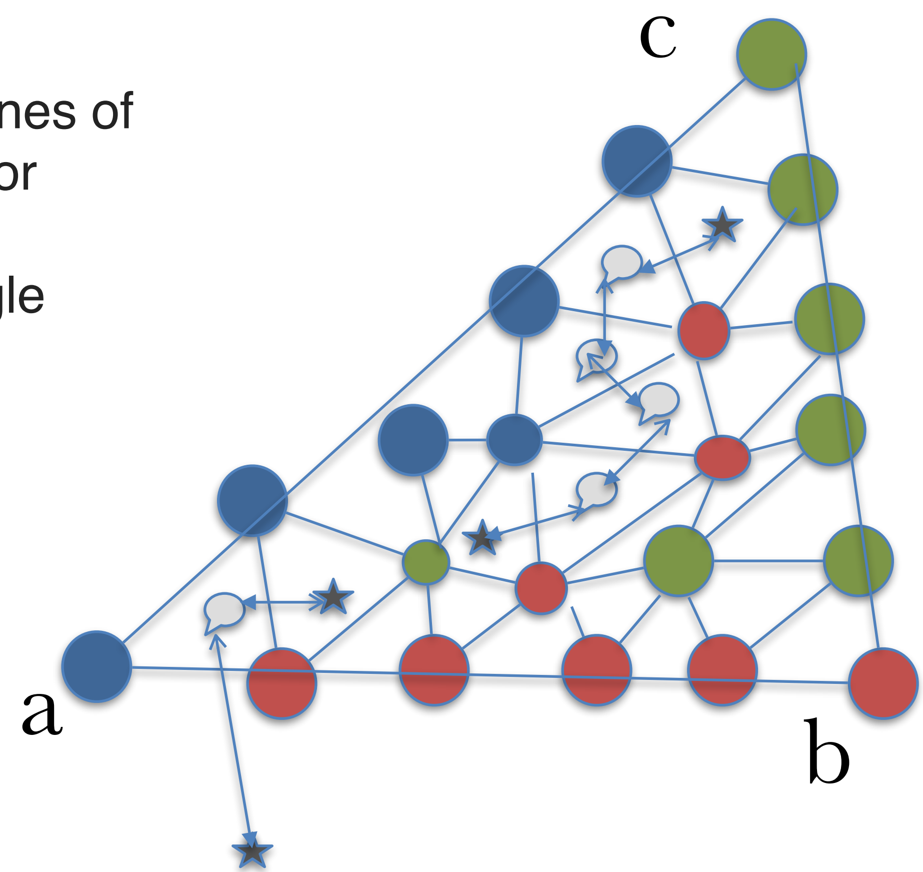
Starting node:
The outside of **SPERNER**

Any other degree one node:
a base triangle of all three colors



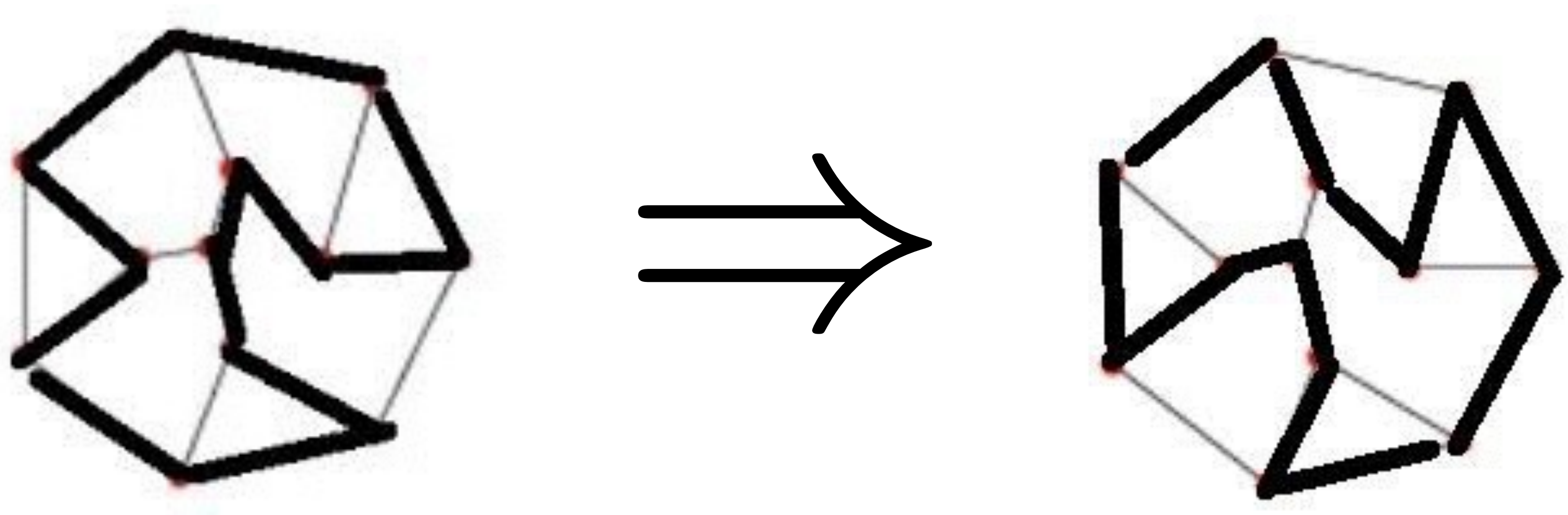
SPERNER

SPERNER: Boundary vertices are so colored that along each of the three lines of the triangle ABC there is only one color internally
Corollary: staring node (outside triangle ABC) is of degree one.

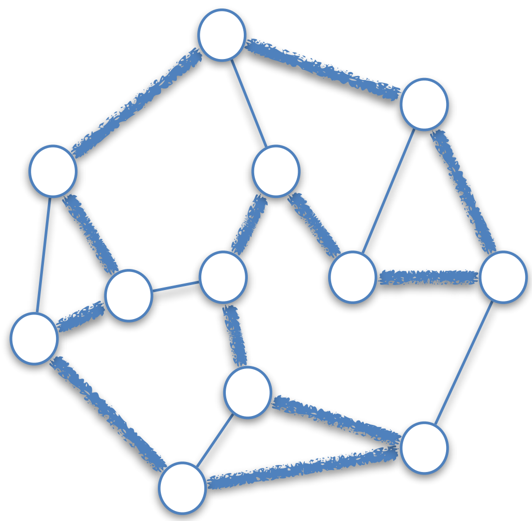


Smith Problem

Given a cubic graph G=(V,E) & given a Hamiltonian cycle H. There is another different Hamiltonian cycle H'.

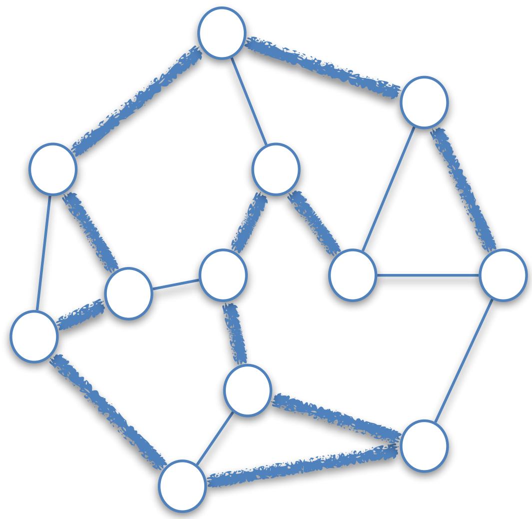


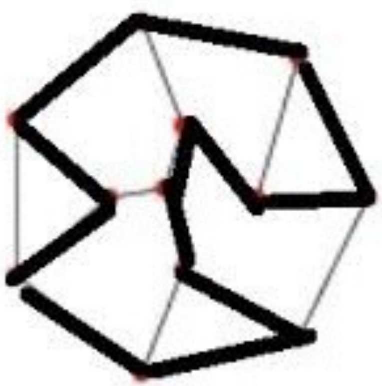
Smith Problem



cycle=>path=>lollipop=>…=>lollipop=>cycle

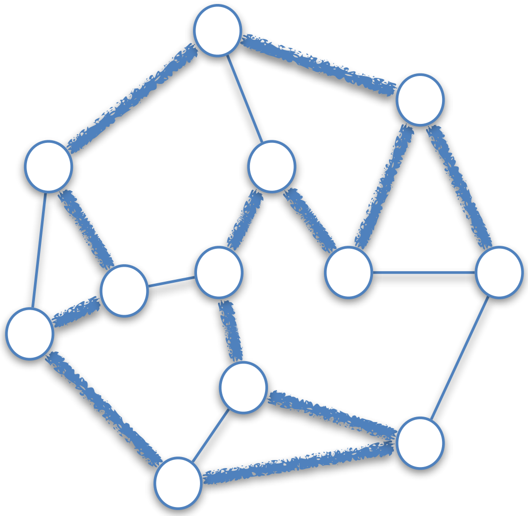
Smith Problem





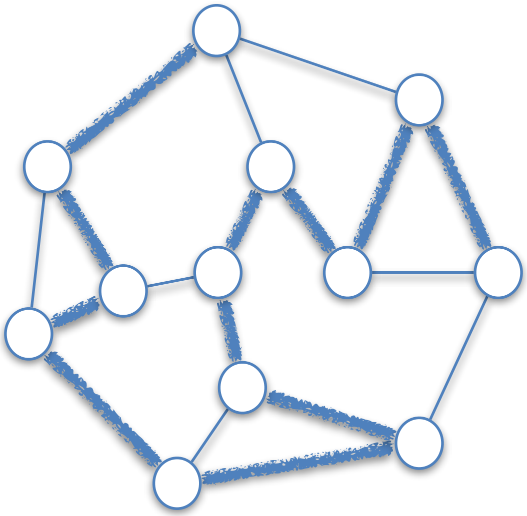
cycle=>path

Smith Problem



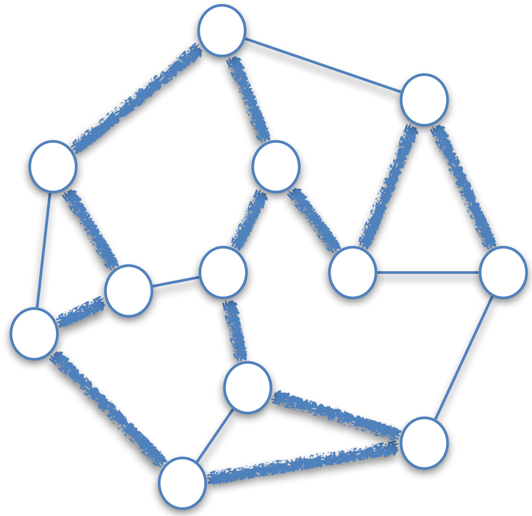
path=>lollipop

Smith Problem



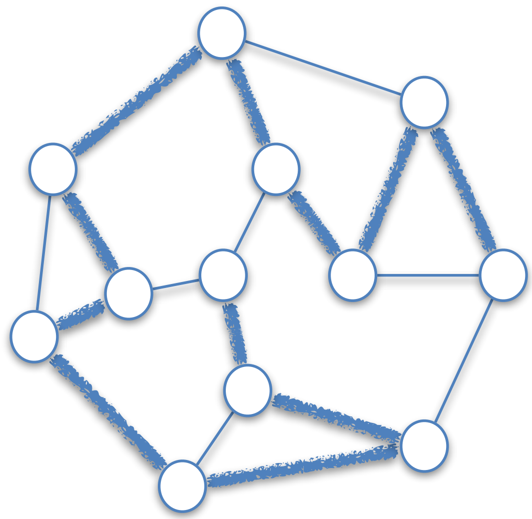
lollipop=>path

Smith Problem



path=>lollipop

Smith Problem



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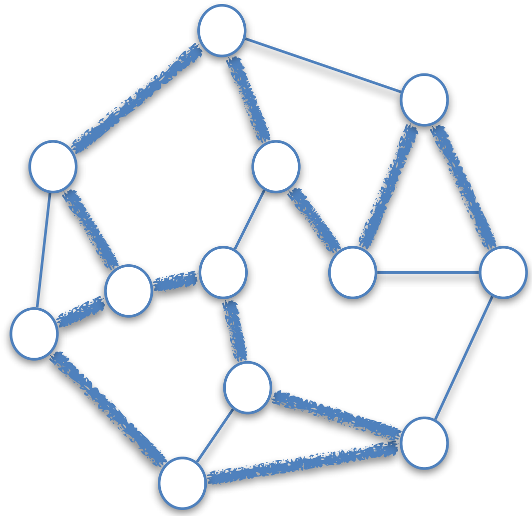
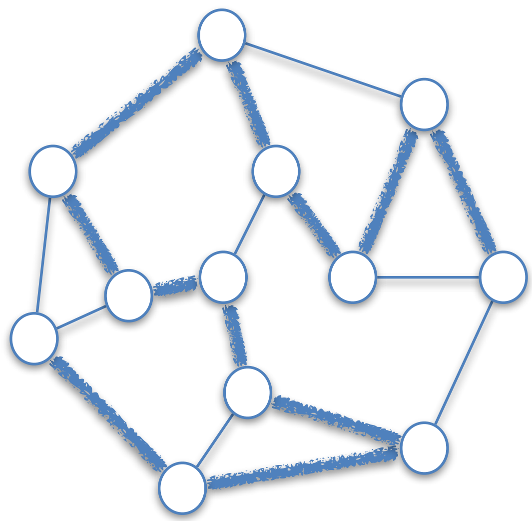
Smith Problem



path=>lollipop

Smith Problem



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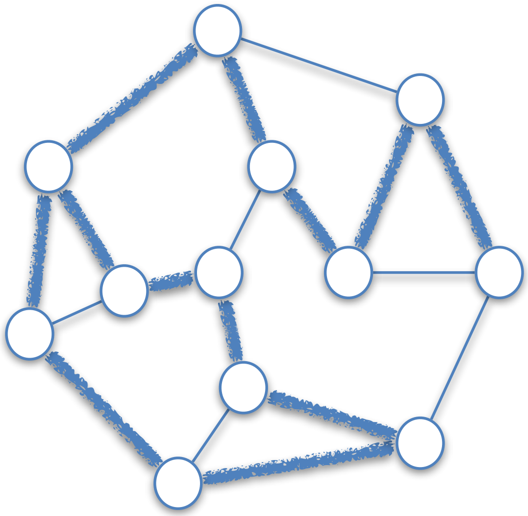
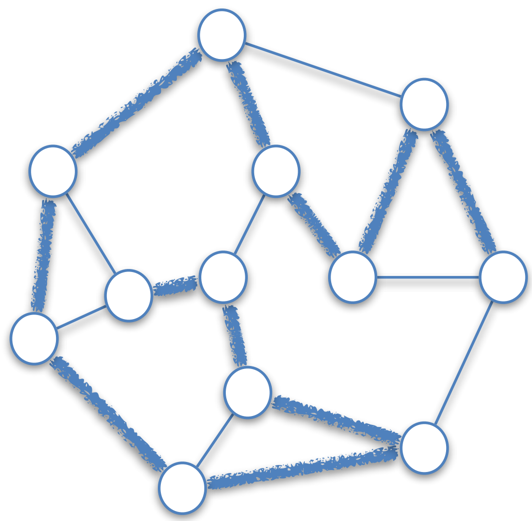
Smith Problem



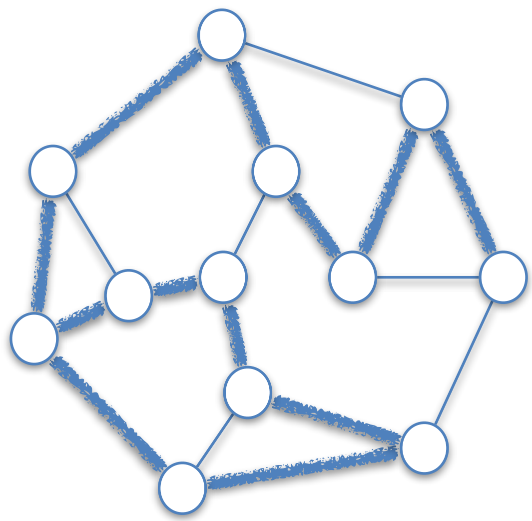
path=>lollipop

Smith Problem



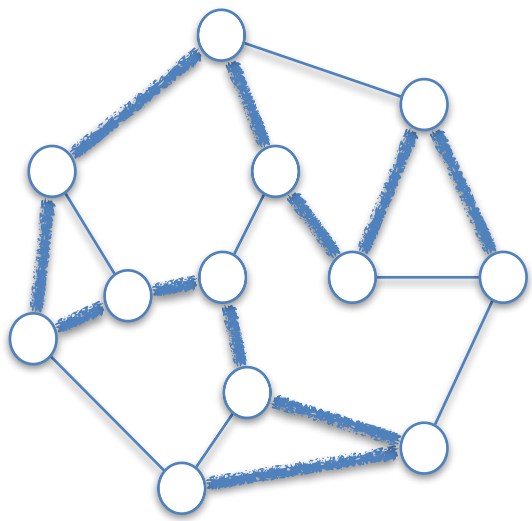
lollipop=>path

Smith Problem



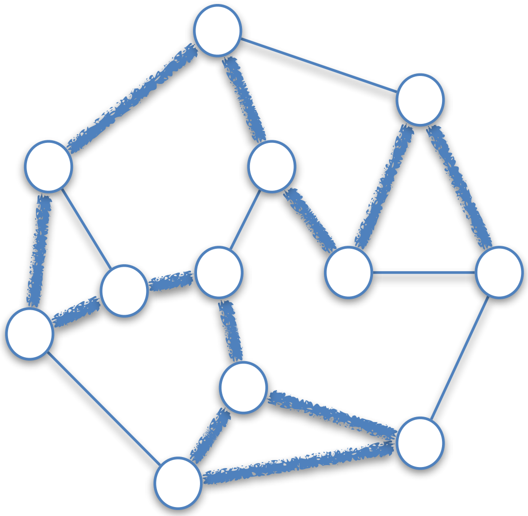
path=>lollipop

Smith Problem



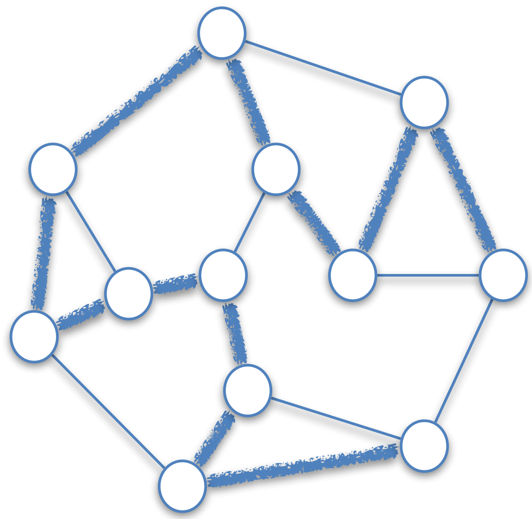
lollipop=>path

Smith Problem



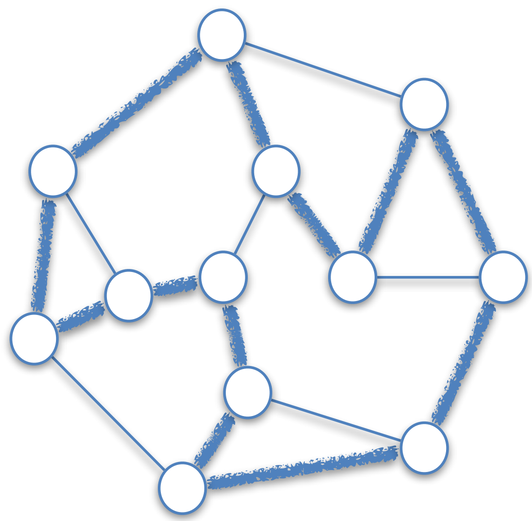
path=>lollipop

Smith Problem

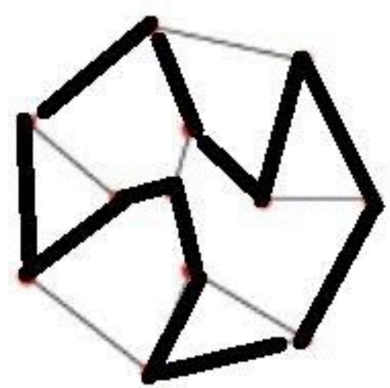


lollipop=>path

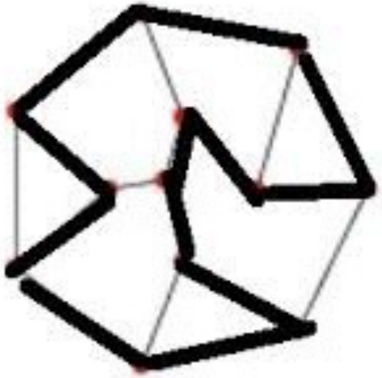
Smith Problem

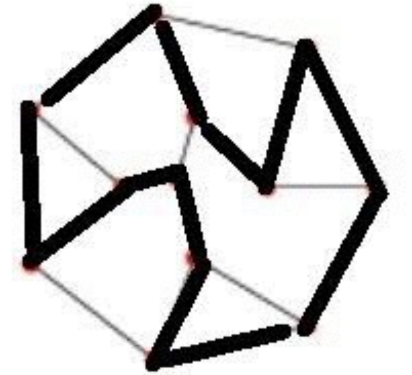


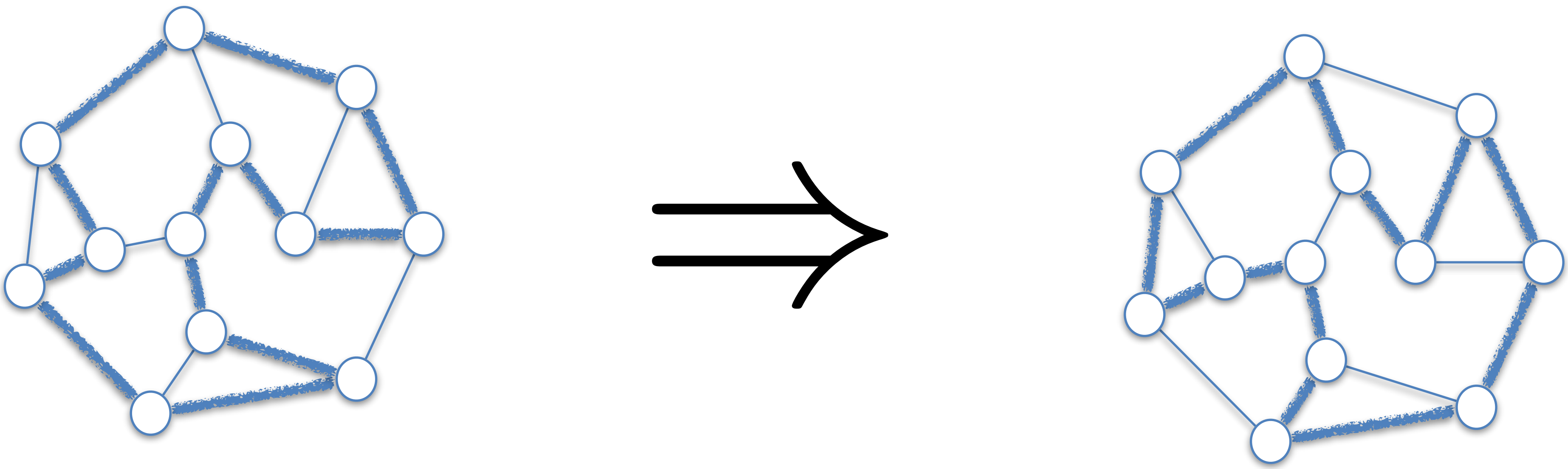
path=>another cycle



Smith Problem







The underlying graph

Nodes: a lollipop or a cycle.

Edges: between two lollipops/cycles linked by a path

Starting node:
The given H-cycle

Any other degree one node:
any other cycle

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Another End of Directed Lines (AEDL)

How to Create Directions?

Requirements:

1. Local Computation Decision
2. Consistency on each path/cycle

Examples:

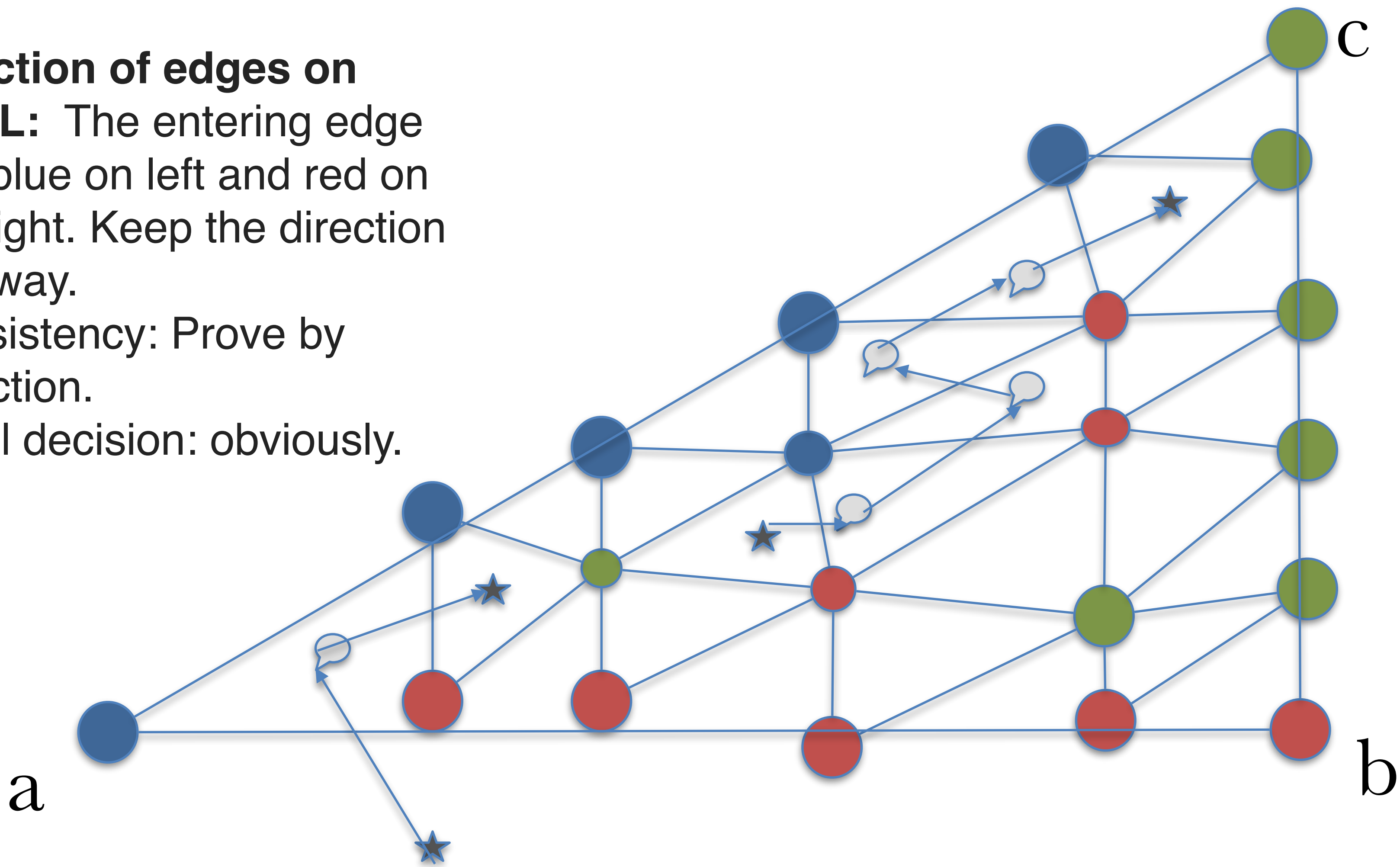
1. Possible: The Sperner Lemma
2. Not till now: Smith Problem

AEDL： Directions in SPERNER Triangulation！

Direction of edges on AEDL: The entering edge has blue on left and red on the right. Keep the direction that way.

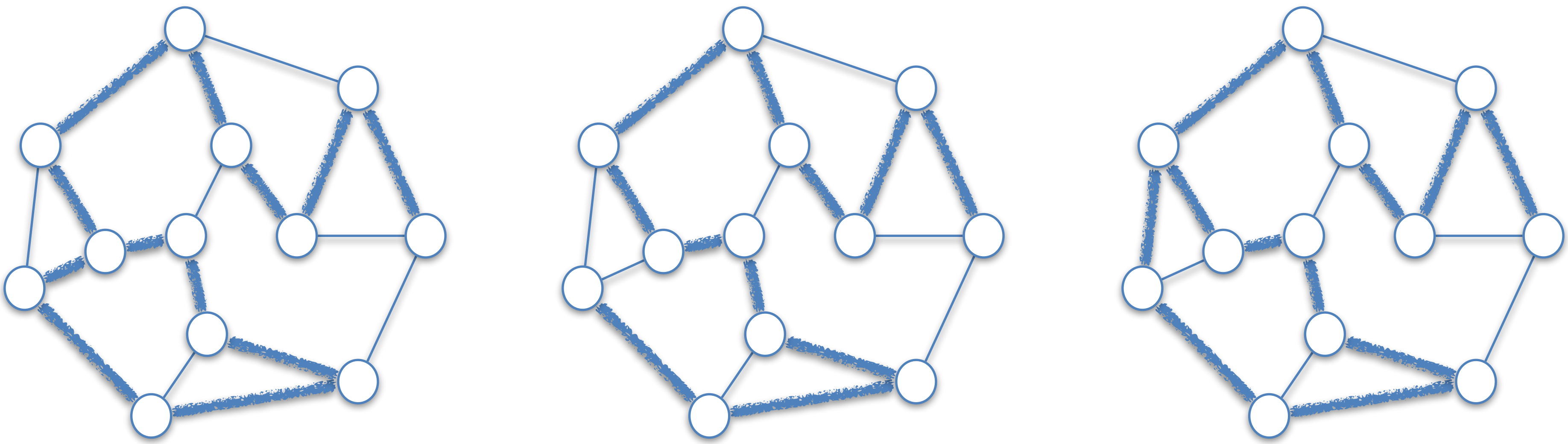
Consistency: Prove by induction.

Local decision: obviously.



Smith Problem

Node- - - - edge- - - - - Node



AEDL：Directions in Smith’s problem?

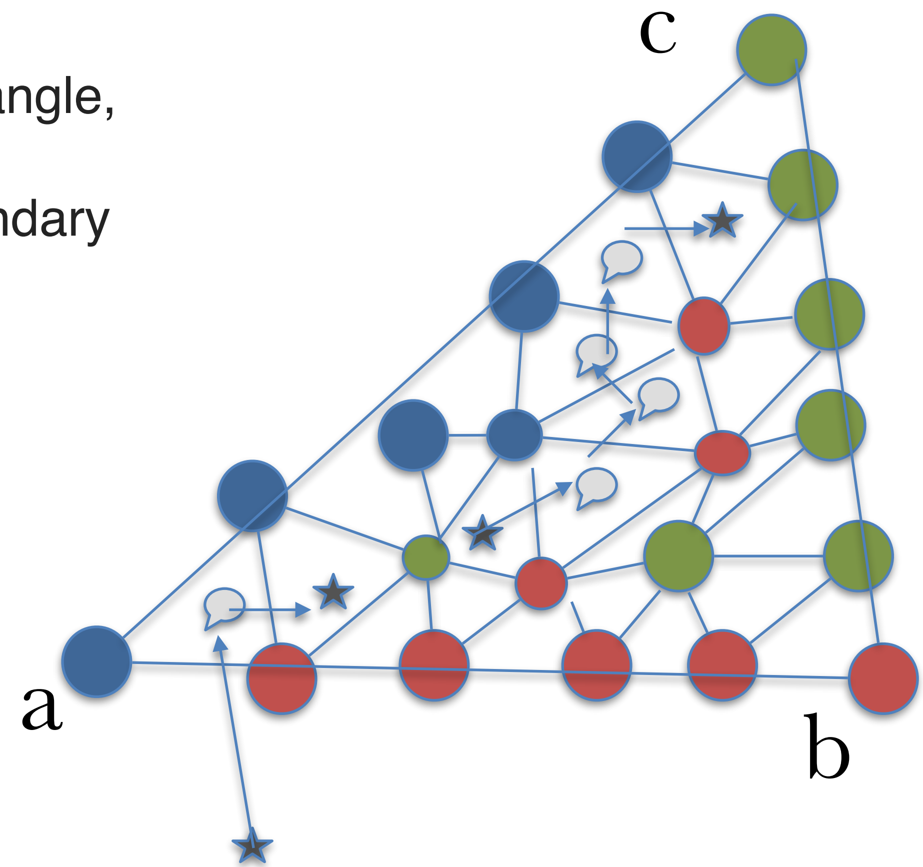
Edge is between two (lollipop/cycle)s add edge on a path

Exactly one possibilities with no direction
No direction can be created at this point?

Assign Direction to **SPERNER**

Direction on SPERNER:
Node set: consisting of each base triangle, and outside triangle region,
Edge set: Two nodes sharing an boundary edge of colors blue and red.

Direction of an edge: chosen



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Reductions for PPA(D)-Completeness

The problem is PPA(D)-hard, if it can solve AE(D/U)L
The Problem is in PPA(D), it is solved by AE(D/U)L

It is PPA(D)-Complete iff it is both above

Reductions for PPA(D)-hardness

The problem is PPA(D)-hard, if it can solve AE(U/D)L

The Problem is in PPA(D), it is solved by AE(U/D)L

It is PPA(D)-Complete iff it is both above

Examples:

1. Reduction of AEDL to *2D SPERNER*
2. Reduction of AEUL to m-**SPERNER**

Reduction of AEDL to 2D SPERNER

Reduction of AEDL to Planar-AEDL

WHY AEDL is not planar?

Reduction of Planar-AEDL to 2D SPERNER

Input Model of of AEDL

Node set: V={0,1,2,…,N-1} where N= 2^n

Edge set: E={e(i,j): for each $i \in V$ } such that

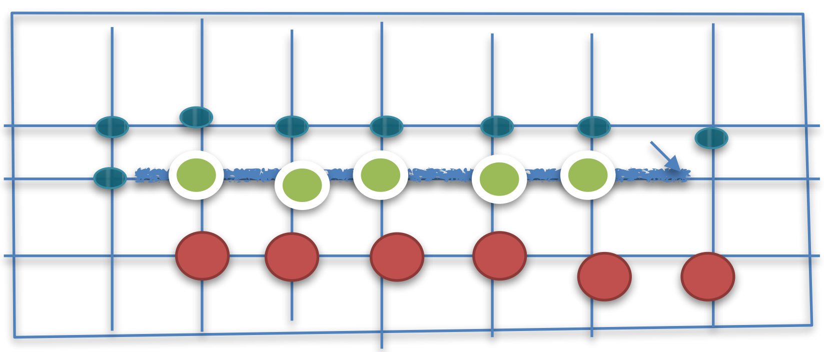
$$0 \leq \delta_-(i), \delta_+(i) \leq 1$$

such that j in e(i,j) is computed in polynomial time.

Planar AEDL reduces to SPERNER

Coloring Scheme:along the direction.
green alone each edge of AEDL
red on left vertices
blue on right vertices

Given starting node: placed at boundary
counter-clockwise direction on boundary red
All other grid points: colored blue



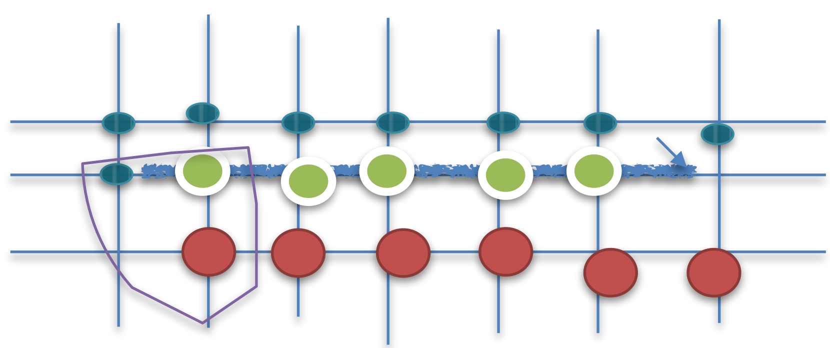
Properties

All Sperner triangle appears at the end of lines of AEDL.

Boundary has one pair of blue-red edge

Sperner solves Planar-AEDL

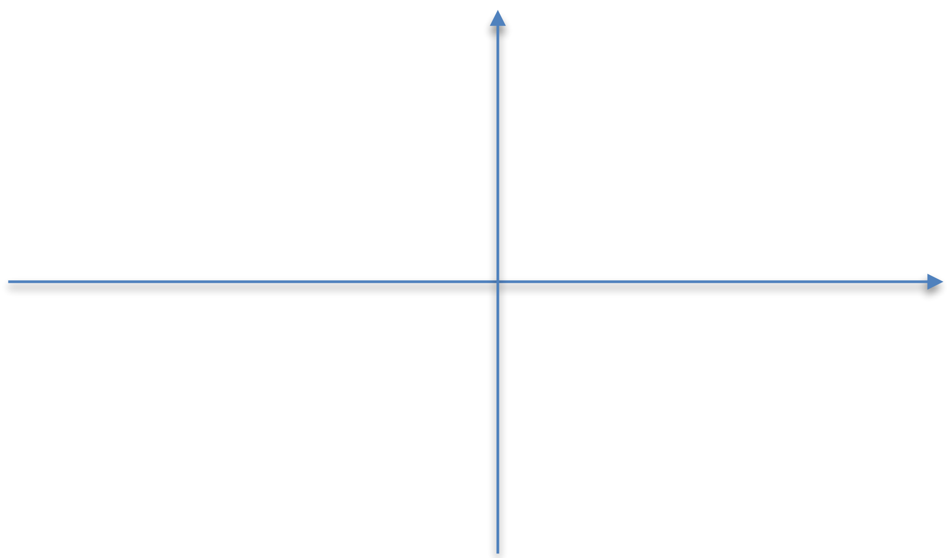
Remaining problem: does not know how to embed lines/cycles in AEDL on the plane in polynomial time (#nodes exponential)



Planar embedding of AEDL

First embed in a fixed way

Then crossing resolution



Planar embedding of AEDL

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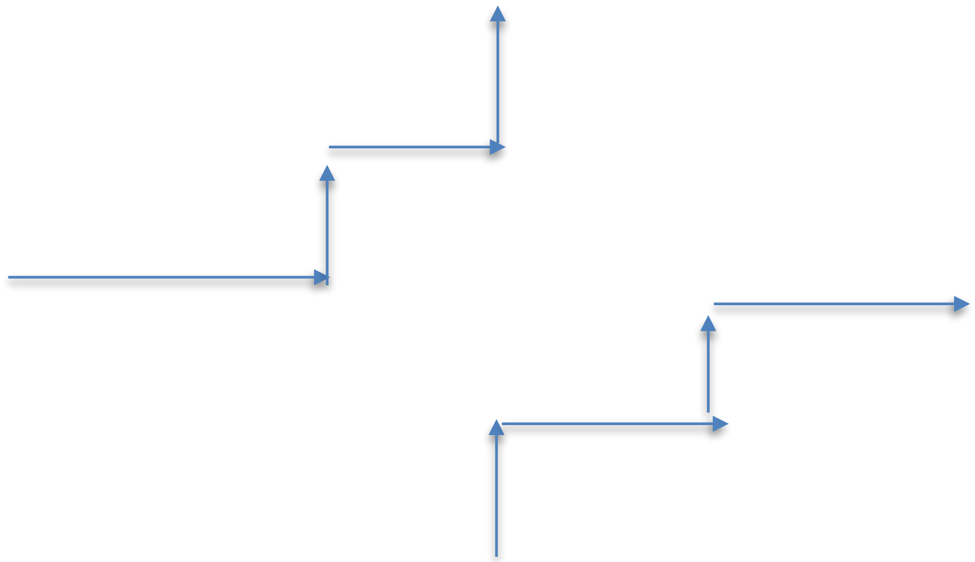
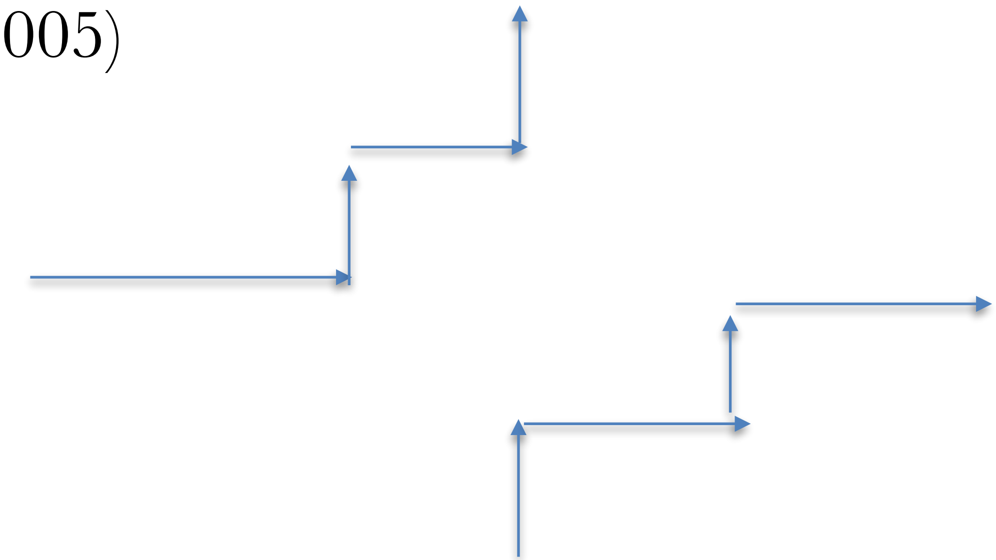


Planar embedding of AEDL

First embed in a fixed way
Then crossing resolution
End of lines preserved.

SPERNER: PPADC (Xi Chen and D, 2005)



Planar embedding of AEUL?

Problem: There is no direction on AEUL

Key Idea：
Create directions, and use AEDL approach
Make reversing lines equivalent with the help of a reversing line on the mobius strip

Created directions in m-SPERNER for AEUL?

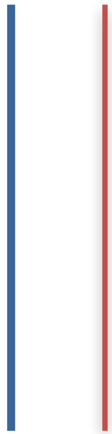
Construction:

1. Create each node as a directed channel, one in another out.
2. For each i with edges (i,j) and (i,k) and j<k, connect in-port to j and outport to k.
    1. difficulty: in-port of i is connected to in-port of j, or out-port of i to the out-port of k.
    2. resolution: use the reversing line of mobius strip
3. Given degree one node placed on the boundary.

Node Channel

Convention: Direction up

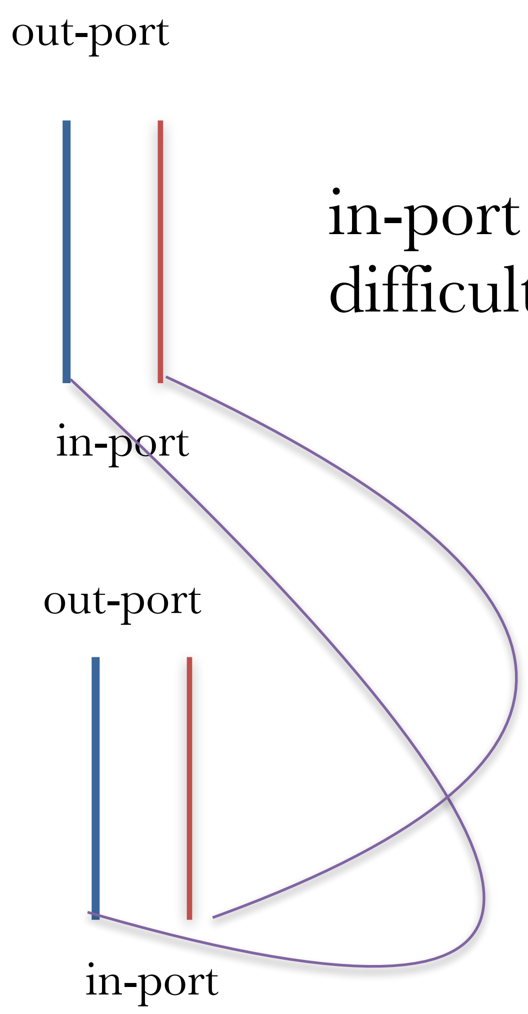
out-port



in-port

Difficult edge connector

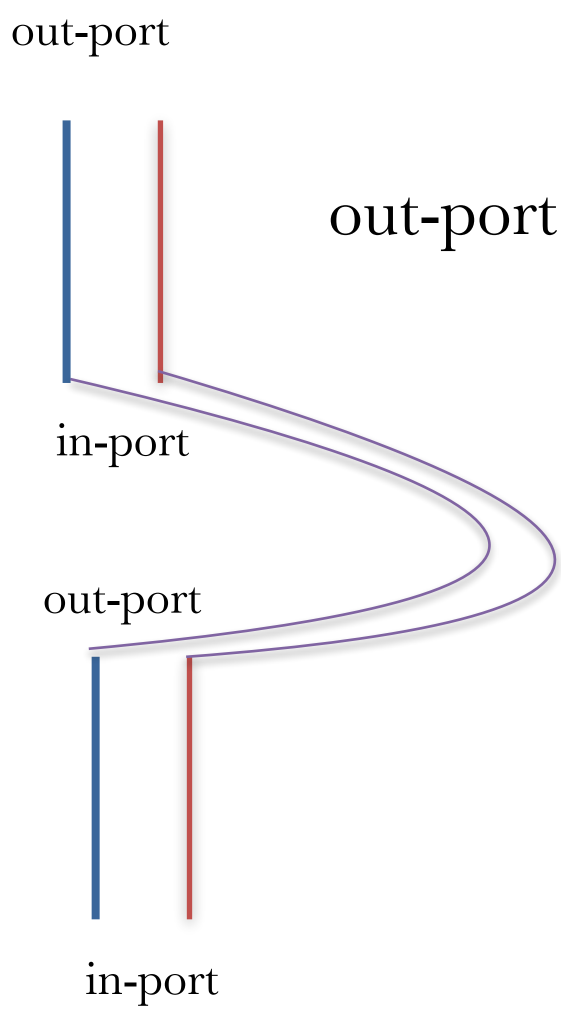
Convention: Direction up



in-port to in-port
difficult self-crossing

Easy edge connector

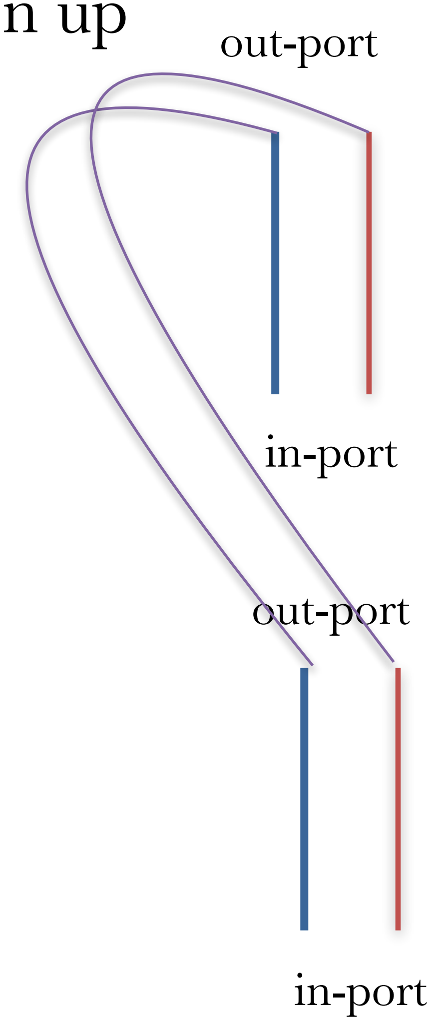
Convention: Direction up



out-port to in-port

Another difficult edge connector

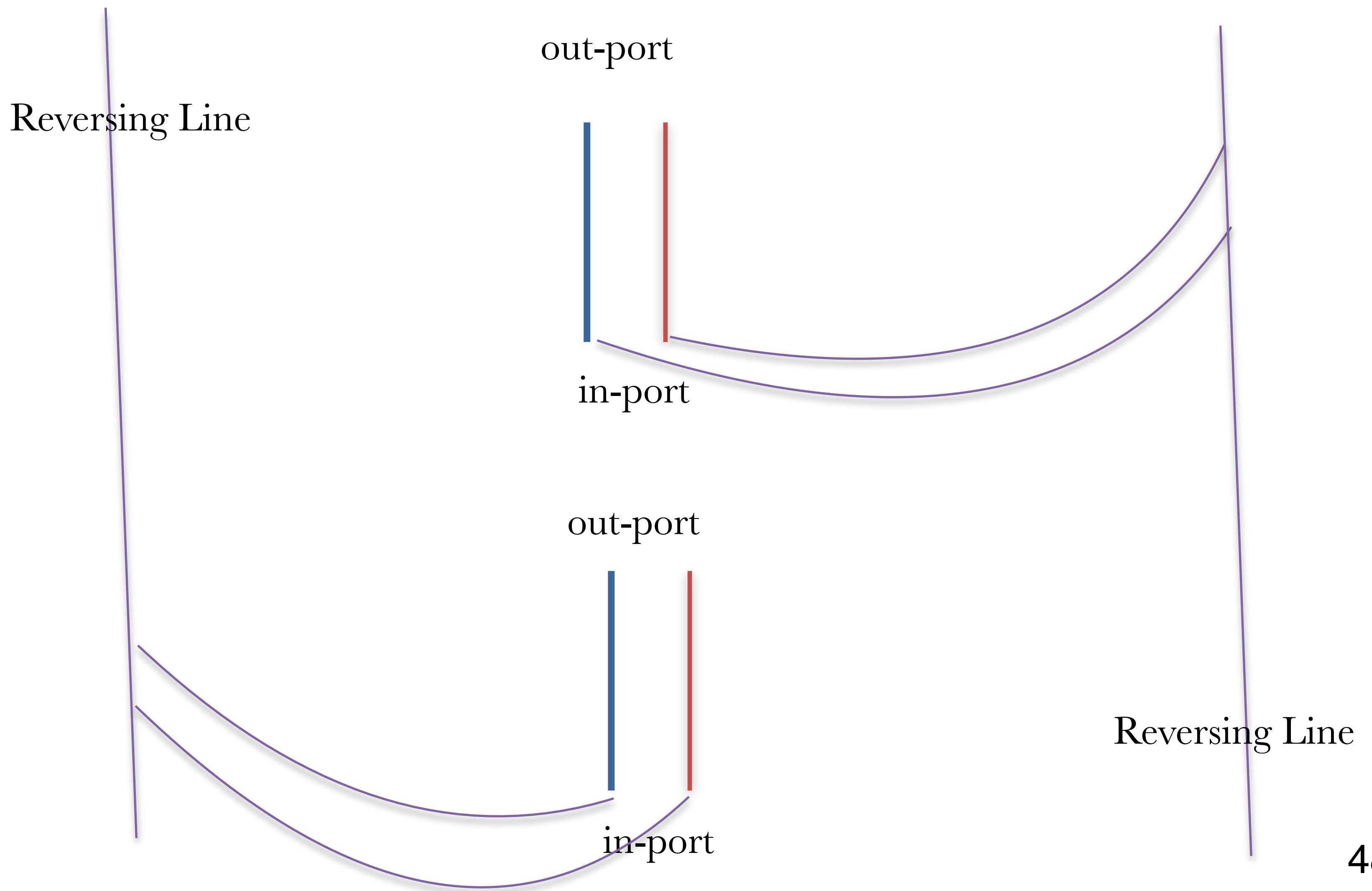
Convention: Direction up



out-port to out-port
difficult self-crossing

Difficult edge connector

With the help of reversing line on Mobius Strip



Mobius strip embedding of AEUL

Construction based on implicit directions (defined by the numeric values of nodes)

Then crossing resolution

End of lines preserved (corresponding to sperner base triangle

m-SPERNER: PPAC (D, Edmonds, Feng, Liu, Qi, Xu, 2015)

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Two Player Nash Equilibrium Solves Fixed Point

Use probability for strategies in 2NASH as numbers/logic_values
Operations on numbers done by probabilities of strategies
Implement SPERNER using Nash

1. Individual operations by 2 players
2. Uniformly distribute probabilities of pairs of strategies
3. Embed (1) many gates to (2) matching penny's game

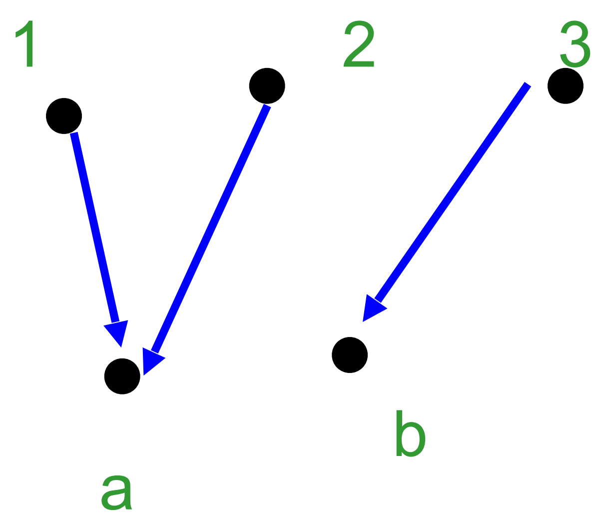
2NASH is PPADC (Xi Chen and D 2006)

Single Gates with 2 Players

- Arithmetic Gates: G_+, G_-, G_c, G_{xc}, $G_=$
- Gate G_+: $v_1+v_2=v_3$
- Player 1 has 3 strategies 1,2,3;2 two a, b
- Value of player 2 depends on probability of player 1's and his own strategies:

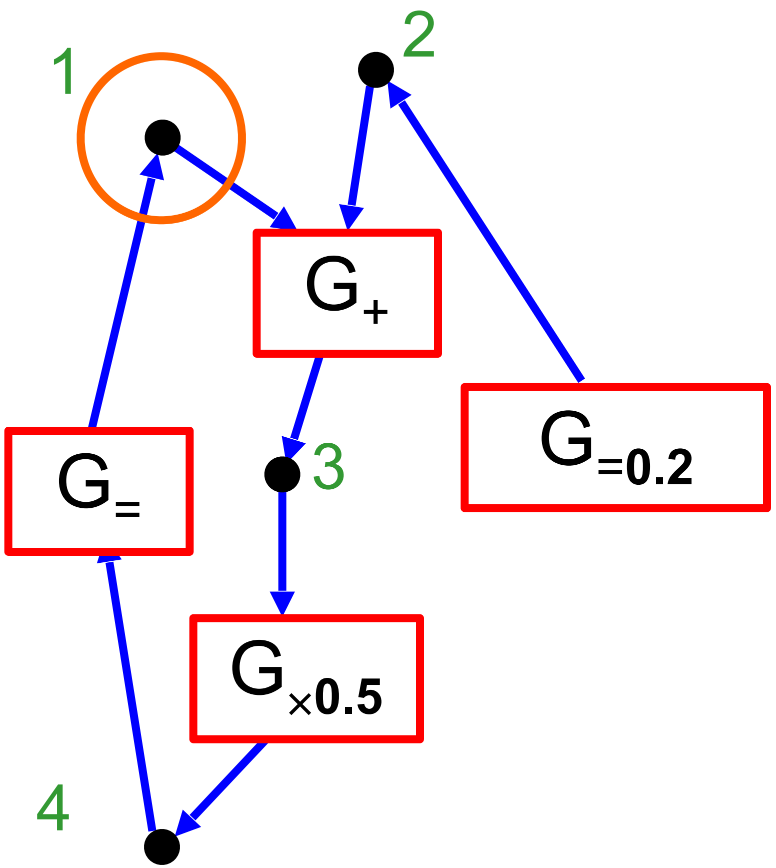
  p(a)*(p(1)+p(2))+p(b)*p(3)

- p(1)+p(2)=p(3) if p(a)*p(b)>0



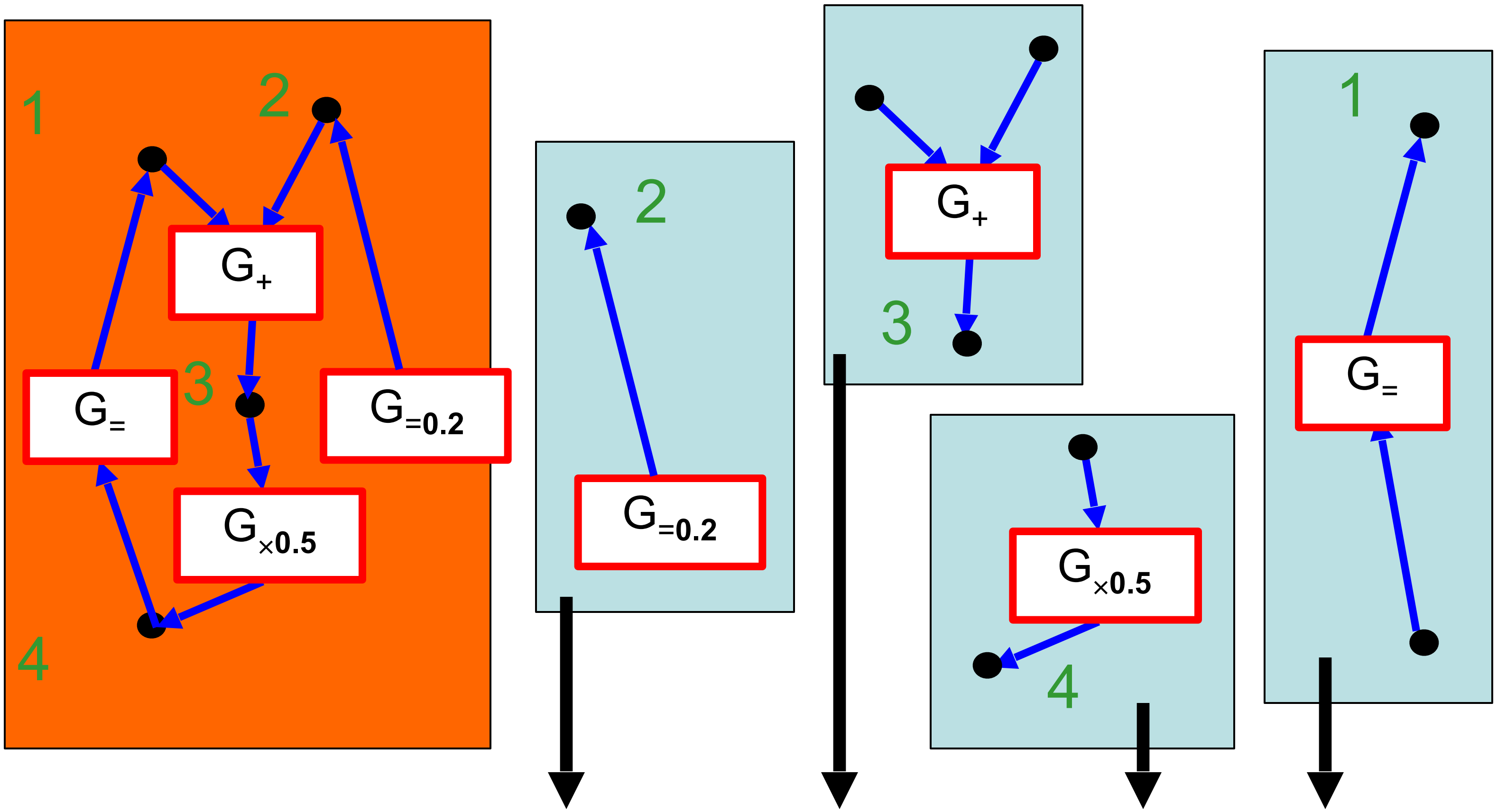
Combined Circuit to Compute Fixed Point

- A set of K Nodes
  - v in [0,1]
- Gates
  - Arithmetic, Logic
- Gate G_+:
  - $v_3 = (v_1+v_2)$
- Rule
- Solution



$(x+0.2)/2=x \quad x=?$

Overview: from GC to 2-Nash



$$(A^*,B^*) + (A_2,B_2) + (A_3,B_3) + (A_4,B_4) + (A_1,B_1)$$

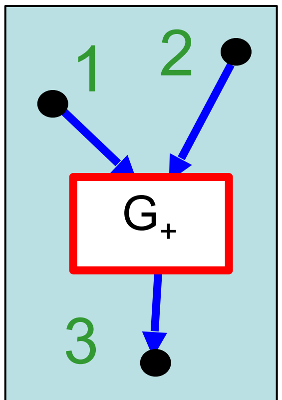
Generalized Matching Pennies

- 2K x 2K, M = 2^K

$$\begin{pmatrix} M & M & 0 & 0 & 0 & 0 & 0 & 0 \\ M & M & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M & M & 0 & 0 & 0 & 0 \\ 0 & 0 & M & M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M & M & 0 & 0 \\ 0 & 0 & 0 & 0 & M & M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M & M \\ 0 & 0 & 0 & 0 & 0 & 0 & M & M \end{pmatrix} \begin{pmatrix} -M & -M & 0 & 0 & 0 & 0 & 0 & 0 \\ -M & -M & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -M & -M & 0 & 0 & 0 & 0 \\ 0 & 0 & -M & -M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -M & -M & 0 & 0 \\ 0 & 0 & 0 & 0 & -M & -M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -M & -M \\ 0 & 0 & 0 & 0 & 0 & 0 & -M & -M \end{pmatrix}$$

- Nash equilibrium: $x_{2i-1}+x_{2i}=y_{2j-1}+y_{2j}=1/K$

Combine many gates to Bimatrix



$$\begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

Row 5 and 6 of A_i Column 5 and 6 of B_i

$$(A^*,B^*) + (A_2,B_2)$$

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Octahedral Tucker

Octahedral Tucker of n dimension:
Side length 2 hyper-grid with vertices colored with

$$\{\pm 1, \pm 2, \cdots, \pm n\}$$

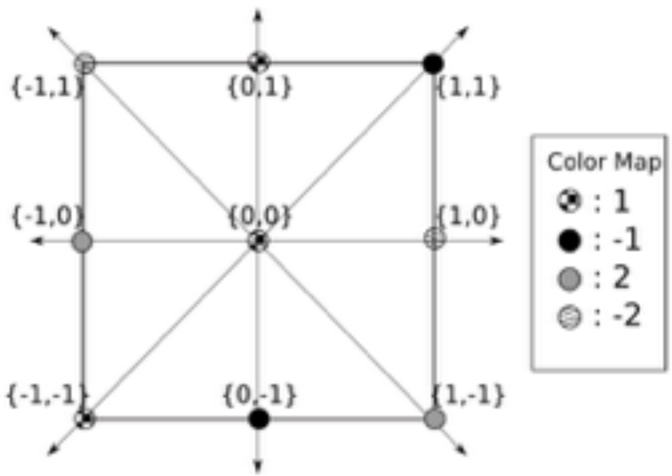
Boundary vertices with antisymmetric colors

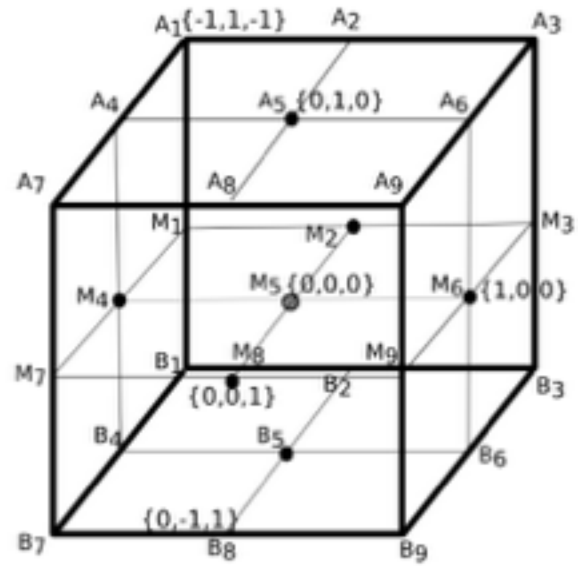
f(p)=-f(-p)

There is an pair of edge complementarily colored: e=(i,j) and f(i)+f(j)=0

Finding one is PPAC (D, Feng, Kulkarni 2017).

Examples in 2D/3D





2x2 Facets:

- $A_1A_3A_9A_7$
- $M_1M_3M_9M_7$
- $B_1B_3B_9B_7$
- $A_1B_1B_7A_7$
- $A_2B_2B_8A_8$
- $A_3B_3B_9A_9$
- $A_1A_3B_3B_1$
- $A_4A_6B_6B_4$
- $A_7A_9B_9B_7$
- $A_1A_7B_9B_3$
- $A_3A_9B_7B_1$
- $A_7A_9B_3B_1$
- $A_1A_3B_9B_7$
- $A_1A_9B_9B_1$
- $A_3B_3B_7A_7$

PPA-Completeness of Octahedral Tucker

From a special sized version of 2D Tucker(proven PPAC)

Reduce one dimension size by half, add a new dimension of size 8.

End at all size 8 dimensions(a polynmial # of them).
Reduce them into size 2

Key requirement:
Size the problem properly
Beat the last step difficulty on a narrow space.

Size the problem properly

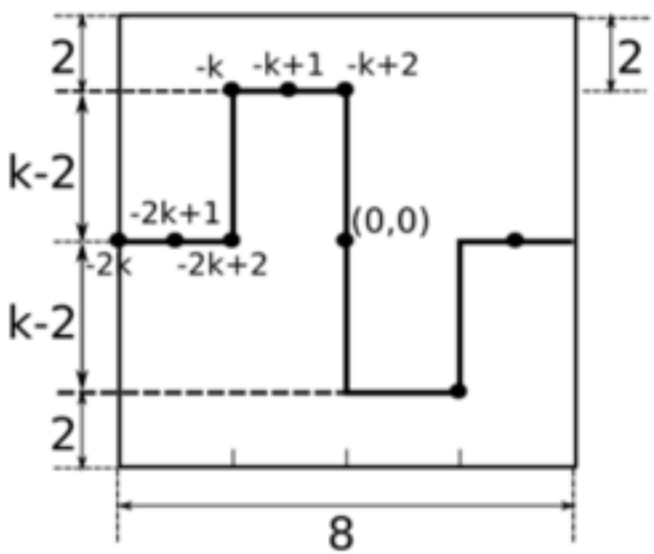
From a special sized version of 2D Tucker.
Make sure it is suitable for all the reductions to follow work well

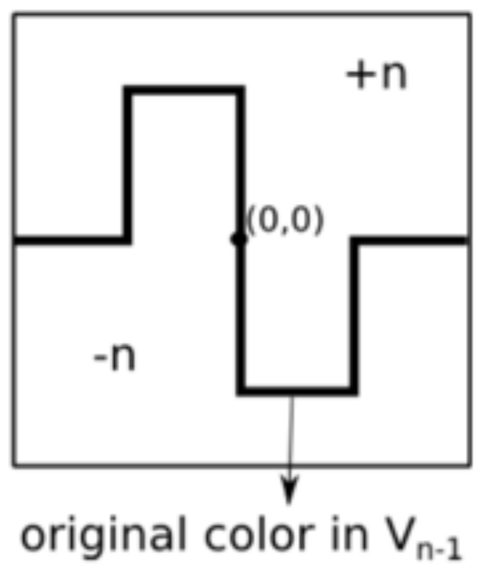
A. Proper size: Derive the starting size of the 2D Tucker problems
B. New triangulations: Make sure octahedral Tucker structure to survive all the subsequent reductions.
C. Create a starting PPAC problem satisfies both conditions

PPA-Completeness of Octahedral Tucker

Make sure reduction is efficient
Not to raise the number of dimensions to become exponential eventually.

A. Reduce one dimension size by half, add a new dimension of size 8.



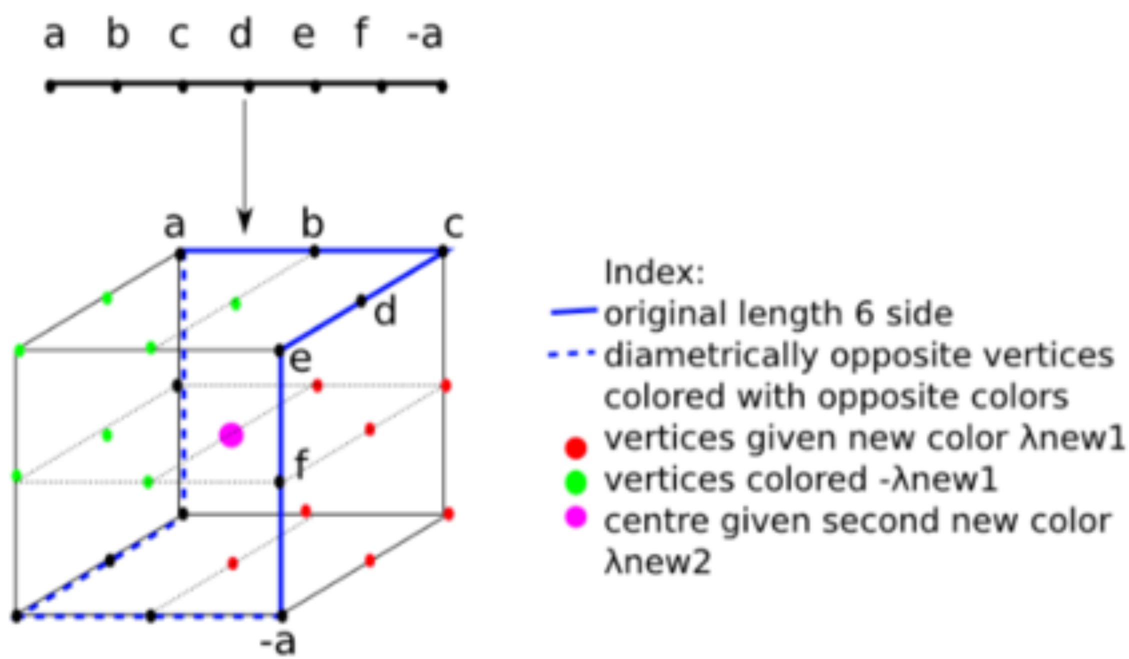


PPA-Completeness of Octahedral Tucker

Reduce all size 8 dimensions(a polynmial # of them) to size 2 at each dimension

Beat the last step difficulty on a narrow space.

An example from size 6 dimension to three each of size 2's



Outline

六、 Challenges

Summary of the Progress

- Computational Equivalence of 2NASH and Fixed Point (class PPAD)
- Mobiles Band Characterization of (class PPA)
- Two Kinds of Fixed Points in Terms of Computation
- **Challenges: PPAC completeness for Related problem in Graphs\Numbers\Combinatorics**

Unbalancedness of Problems in PPA and PPAC

- There has been a tradition of research in PPA problems
- But almost none (actually two) PPAC problems till recently
- There are a lot of known PPAD-complete problem as well as many in PPAD

PPA-Complete Problems

1. Grigni (2001) 3D non-orientalbe space PPAC
2. Fried et al (Grigni 2006) locally 2D space is PPAC
3. D, Edmonds, Feng et al. (2015), 2D m-SPERNER PPAC
4. Assinberger, et al., (2015) 2D TUCKER PPAC
5. Belovs, et al., (2017): PPA-Circuit CNSS and PPA-Circuit Chevalley are PPAC
6. D, Feng, Kulkarni (2017): Octahedral Tucker is PPA-Complete

- Kintali (2009) already compiled a list of 25 PPAD-complete problems; the list is far from complete.

Problems in PPA

A. Papadimitriouc(1991), Beame, Cook, Edmonds, et al.(1998)
   - Smith and Hamiltonian decomposition, Necklace splitting and Discrete Ham sandwich, Explicit Chevalley

B. Cameron and Edmonds (1990,1999)
   - Many graph problems: room partitioning, perfect matching,

C. Jeˇr´abek (2016)
   - square root computation and finding quadratic nonresidues modulo n, into PPA
   - Factoring in PPA under randomized reduction.

D. D, Feng, Papadimitriou (2016): 2D m-TUCKER is in PPA

Thank you!