On some graph modification problems

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ECCO 2017

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- Is it possible to make a graph bipartite with at most k edge contractions? [Heggernes et al., 2013]

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- Identifying such sets may have some nice applications.

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  - Attacker's objective: delete a small set of vertices to restrict the size of a largest cohesive cluster in the remaining graph.
  - Defender's objective: identify a set of vertices whose deletion would substantially decrease the size of largest cohesive cluster, in order to protect these vertices.

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- VERTEX COVER: given a graph G and an integer k, does G contain a subset V' of at most k vertices such that each edge has at least one endvertex in V'?
  - for triangle-free graphs: $\pi = \omega$, $S = \{\text{vertex deletion}\}$, $d = 1$.

π and S

- Graph parameters considered so far:
  - independence number α;
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  - independence number α;
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  - matching number μ.
- Set S always consisted of a single operation:
  - vertex deletion;
  - edge deletion;
  - edge addition.

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Theorem [Bentz et al., 2010]

Even if d is fixed, the problem is NP-complete in bipartite graphs. But it is polynomial-time solvable in grid graphs and graphs of bounded treewidth.

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Theorem [Bazgan et al., 2015]

The problem is polynomial-time solvable in threshold graphs. If d is fixed it is polynomial-time solvable in split graphs.

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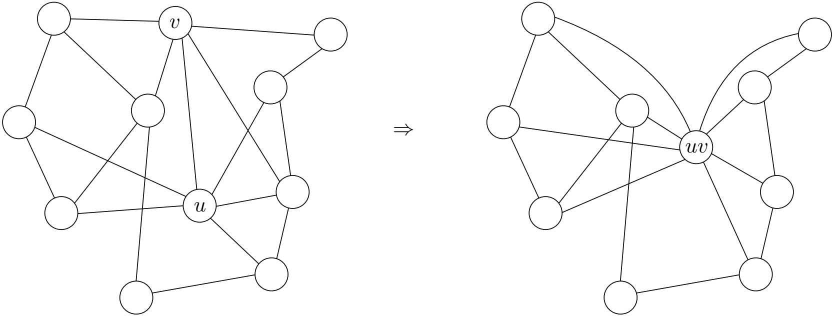

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Let G, G' be two graphs. We say that G can be k-contracted into the graph G' if G can be modified into G' by a sequence of at most k edge contractions.

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PROBLEMS WITH $\pi \in \{\alpha, \omega, \chi\}$

CONTRACTION BLOCKER(π)
Instance: a graph G and two integers $d, k \geq 0$
Question: can G be k-contracted into G' with $\pi(G') \leq \pi(G) - d$?

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- If d is not part of the input, but fixed instead, the corresponding problems will be denoted by:
  - d-CONTRACTION BLOCKER(π)
  - d-DELETION BLOCKER(π)

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- Notice that bipartite graphs are not closed under edge contraction; therefore membership to NP cannot be established.

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- An edge contraction does not increase α or μ.
- Since $\alpha(T) + \mu(T) = n$ and $\alpha(T') + \mu(T') = n - 1$, it follows that:
  - either $\alpha(T') = \alpha(T) - 1$,
  - or $\mu(T') = \mu(T) - 1$.

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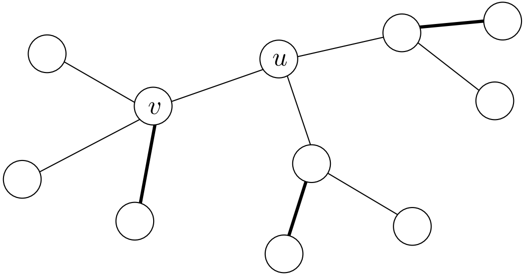

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  - $\Rightarrow$ if $k \geq d$, the answer is YES, otherwise the answer is NO.

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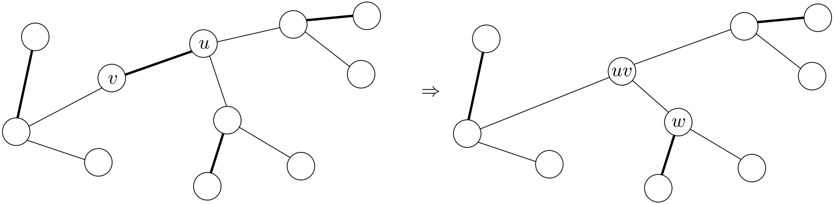

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  - We can show that this is optimal.
  - $\Rightarrow$ if $k \geq 2(d + \mu(T)) - n$, the answer is YES, otherwise it's NO.

TREES - BIPARTITE GRAPHS

- We have

$$\mu(G) + \alpha(G) = n$$

for any bipartite graph G!

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- Being closed under edge contraction plays an important role in our proof!

Theorem [Costa et al., 2011], [Bazgan et al., 2011]

For $\pi \in \{\alpha, \omega, \chi\}$, $\texttt{DELETION BLOCKER}(\pi)$ is polynomial-time solvable in bipartite graphs.

INTERVAL GRAPHS

Interval graph

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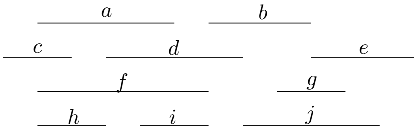

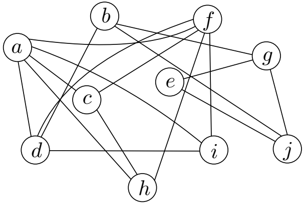

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- Two technical Lemmas are needed as well.

INTERVAL GRAPHS

Theorem [Dinner, Paulusma, Picouleau, R., 2015]

Let $\pi \in \{\omega, \chi\}$. Then $\texttt{CONTRACTION BLOCKER}(\pi)$ can be solved in polynomial time on interval graphs.

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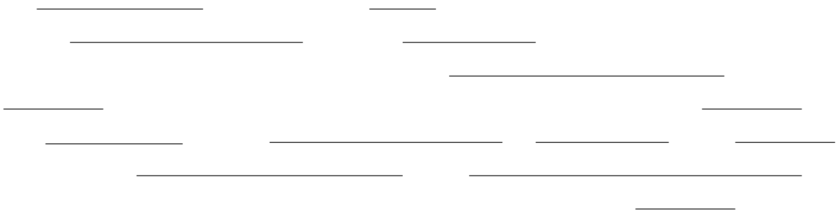

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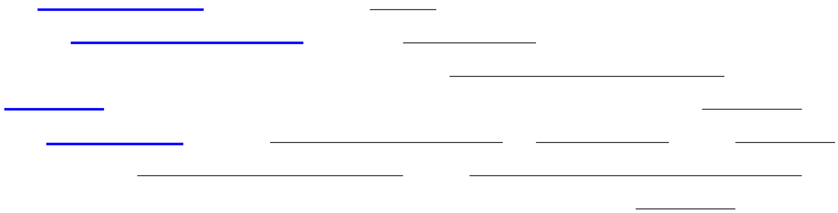

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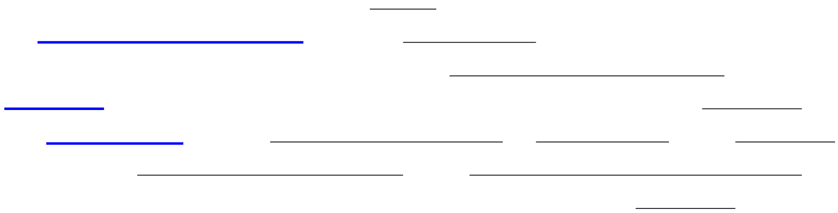

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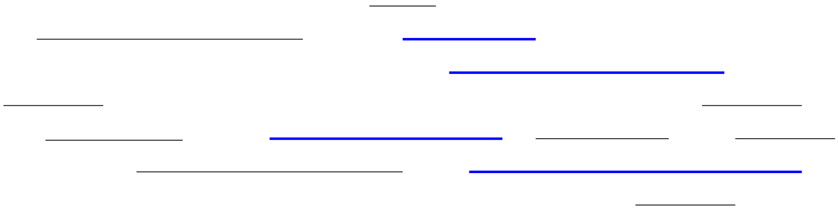

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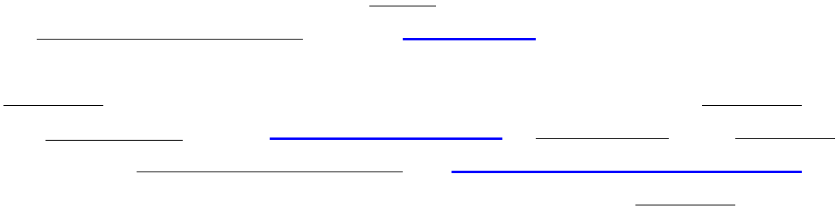

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Theorem [Dinner, Paulusma, Picouleau, R., 2015]

Let $\pi \in \{\omega, \chi\}$. Then `CONTRACTION BLOCKER(`π`)` can be solved in polynomial time on interval graphs.

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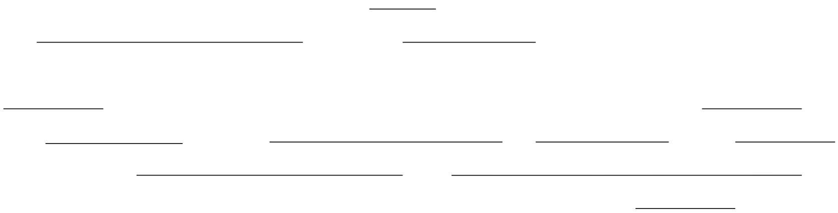

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Theorem [Dinner, Paulusma, Picouleau, R., 2015]

Let $\pi \in \{\omega, \chi\}$. Then `CONTRACTION BLOCKER`(π) can be solved in polynomial time on interval graphs.

Using similar arguments, we can also show the following:

Theorem [Dinner, Paulusma, Picouleau, R., 2015]

Let $\pi \in \{\omega, \chi\}$. Then `DELETION BLOCKER`(π) can be solved in polynomial time on interval graphs.

CHORDAL GRAPHS

The previous results cannot be generalized to chordal graphs:

Theorem [Paulusma, Picouleau, R., 2016]

Let $\pi \in \{\omega, \chi\}$. Then 1-CONTRACTION BLOCKER(π) and 1-DELETION BLOCKER(π) are NP-complete for chordal graphs.

CHORDAL GRAPHS

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- For $\pi = \alpha$, the complexity of both problems, CONTRACTION BLOCKER(α) and DELETION BLOCKER(α), is unknown in interval graphs.
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A graph G is a cograph if it does not contain any path on 4 vertices as an induced subgraph.

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A graph G is a cograph if it does not contain any path on 4 vertices as an induced subgraph.

Theorem [Dinner, Paulusma, Picouleau, R., 2015]

For $\pi \in \{\alpha, \omega, \chi\}$, both CONTRACTION BLOCKER(π) and DELETION BLOCKER(π) can be solved in polynomial time for cographs.

SPLIT GRAPHS

Split graph

A graph G is a split graph if its vertex set can be partitioned into a clique and a stable set.

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A graph G is a split graph if its vertex set can be partitioned into a clique and a stable set.

Theorem [Costa et al., 2011]

For $\pi \in \{\alpha, \omega, \chi\}$, DELETION BLOCKER($\pi$) is NP-complete and d-DELETION BLOCKER(π) is polynomial time solvable for split graphs.

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SUBCLASSES OF PERECT GRAPHS

	CONTRACTION BLOCKER(π)		DELETION BLOCKER(π)	
Class	$\pi = \alpha$	$\pi = \omega = \chi$	$\pi = \alpha$	$\pi = \omega = \chi$
Trees	P	P	P*	P
Bipartite	NP-h	P	P*	P
Cobipartite	$d = 1$: NP-c	NP-c d fixed: P	P	P*
Cograph	P	P	P	P
Split	NP-c d fixed: P	NP-c d fixed: P	NP-c* d fixed: P*	NP-c* d fixed: P*
Interval		P		P
Chordal	NP-c	$d = 1$: NP-c	NP-c	$d = 1$: NP-c
Perfect	$d = 1$: NP-h	$d = 1$: NP-h	NP-c	$d = 1$: NP-c

P_ℓ-FREE GRAPHS

- Cographs are equivalent to P_4-free graphs, i.e. graphs not containing an induced path on 4 vertices.

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- So we obtain the following dichotomy from the previous results for P_ℓ-free graphs, i.e. not containing any induced path on ℓ vertices.

Theorem [Dinner, Paulusma, Picouleau, R., 2015]

Let $\pi \in \{\alpha, \omega, \chi\}$. Then CONTRACTION BLOCKER(π) and DELETION BLOCKER(π) can be solved in polynomial time in P_ℓ-free graphs if $\ell \leq 4$ and NP-hard if $\ell \geq 5$.

1. INTRODUCTION

2. PREVIOUS WORK

3. VERTEX DELETION, EDGE CONTRACTION, $\pi \in \{\alpha, \omega, \chi\}$
   - SUBCLASSES OF PERFECT GRAPHS
   - *H*-FREE GRAPHS

4. CONCLUSION

DEFINITIONS

H-free graph

Let G, H be two graph. Then G is said to be H-free, if it does not contain H as an induced subgraph.

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- If H is an induced subgraph of G, we write $H \subseteq_i G$.
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clique-proof

We say that a graph class $\mathcal{G}$ is clique-proof if

$$G \in \mathcal{G} \implies G \oplus G \in \mathcal{G}, \; G \oplus K_s \in \mathcal{G}$$

SOME USEFUL RESULTS

Theorem [Paulusma, Picouleau, R., 2016]

If `CLIQUE` is NP-complete for a clique-proof graph class $\mathcal{G}$, then `CONTRACTION BLOCKER`(ω) is co-NP-hard for $\mathcal{G}$, even if $d = k = 1$.

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Theorem [Král et al., 2011]

Let H be a graph. If $H \subseteq_i P_4$ or $H \subseteq_i P_3 \oplus K_1$, then the COLORING problem is polynomial-time solvable for H-free graphs, otherwise it is NP-hard for H-free graphs.

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Theorem [Paulusma, Picouleau, R., 2017]

1-DELETION BLOCKER(α), with $k = 1$, is NP-hard even for graphs of girth at least g, for every fixed $g \geq 3$.

H-FREE GRAPHS, $\pi \in \{\omega, \alpha\}$

Theorem [Dinner, Paulusma, Picouleau, R., 2015-2017]

Let H be a graph. Then the following holds:

- If $H \subseteq_i P_4$, then DELETION BLOCKER(α) and CONTRACTION BLOCKER(α) are polynomial-time solvable for H-free graphs; otherwise both are NP-hard or co-NP-hard for H-free graphs.
- If $H \subseteq_i P_4$, then DELETION BLOCKER(ω) is polynomial-time solvable for H-free graphs; otherwise it is NP-hard or co-NP-hard for H-free graphs.
- Let $H \neq K_3 \oplus K_1$. If $H \subseteq_i P_4$ or $H \subseteq_i$ paw, then CONTRACTION BLOCKER(ω) is polynomial-time solvable for H-free graphs; otherwise it is NP-hard or co-NP-hard for H-free graphs.

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  - If r is even, H either contains C_4 or $2P_2$; use previous result on split graphs.

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  - If H contains $3P_1$, use previous result on cobipartite graphs.
  - So suppose that H is $3P_1$-free; hence it must contain $2P_2$ otherwise $H \subseteq_i P_4$; use previous result on split graphs.

□

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  - If $r = 3$, use previous result on graphs of girth at least $g = 4$.

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  - If $2P_2 \subseteq_i H$, use previous result on split graphs.

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- So we may assume now that H is a forest.
  - If $2P_2 \subseteq_i H$, use previous result on split graphs.
  - If $3P_1 \subseteq_i H$, we use the fact the DELETION BLOCKER(ω) is NP-hard for triangle-free graphs. ($\Rightarrow$ VERTEX COVER)

H-FREE GRAPHS, $\pi = \chi$

Theorem [Dinner, Paulusma, Picouleau, R., 2015-2017]

Let H be a graph. Then the following holds:

- If $H \subseteq_i P_4$, then CONTRACTION BLOCKER(χ) is polynomial-time solvable for H-free graphs; otherwise it is NP-hard for H-free graphs.
- If $H \subseteq_i P_4$ or $H \subseteq_i P_3 \oplus K_1$, then DELETION BLOCKER($\chi$) is polynomial-time solvable for H-free graphs; otherwise it is NP-hard or co-NP-hard for H-free graphs.

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4. CONCLUSION

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- We presented blocker problems and their relations with other graph problems.
- In particular we focused on the following two problems and presented results for subclasses of perfect graphs and for H-free graphs.

CONTRACTION BLOCKER(π)

Instance: a graph G and two integers $d, k \geq 0$

Question: can G be k-contracted into G' with $\pi(G') \leq \pi(G) - d$?

DELETION BLOCKER(π)

Instance: a graph G and two integers $d, k \geq 0$

Question: can G be k-vertex-deleted in G' with $\pi(G') \leq \pi(G) - d$?

OPEN PROBLEMS

What is the complexity of

- (1-)CONTRACTION BLOCKER(α) and (1-)DELETION BLOCKER(α) for interval graphs?
- 1-CONTRACTION BLOCKER(α) and 1-DELETION BLOCKER(α) for chordal graphs?
- CONTRACTION BLOCKER(ω) for $(K_3 \oplus K_1)$-free graphs?
- 1-CONTRACTION BLOCKER(π) and 1-DELETION BLOCKER(π) for H-free graphs with $k = 1$ and $\pi \in \{\omega, \alpha\}$?

Thank you for your attention!